Lecture 18

Recall, an integral is the signed area under a curve.

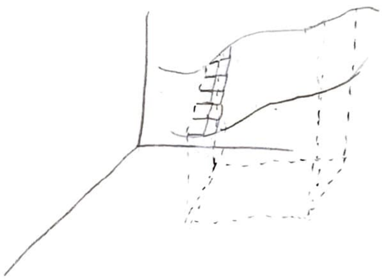
Recall also the we rigorously defined the integral using a Riemann sum.

The Double Integral

We define the double integral as

$$\iint_R f(x,y)\,dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{N} f(x_i, y_j)\,\Delta A$$

this is the double Riemann sum



lots of parallelepipeds.

Properties

$$\iint_R \left[f(x,y) + g(x,y)\right] dA = \iint f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint c\,f(x,y)\,dA = c\iint f(x,y)\,dA$$

Ex. 1

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx$$

$$\int_0^3 \left[\int_1^2 x^2 y\,dy\right] dx$$

$$\int_1^2 x^2 y\,dy = x^2 \int_1^2 y\,dy = \frac{1}{2} x^2 \left[y^2\right]\Big|_1^2 = \frac{1}{2} x^2 (4-1) = \frac{3}{2} x^2$$

$$\int_0^3 \frac{3}{2} x^2\,dx = \frac{3}{2}\frac{1}{3} x^3 \Big|_0^3 = \frac{27}{2}$$

Ex. 2

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy$$

$$y \int_0^3 x^2 \, dx = \frac{1}{3} y \, x^3 \Big|_0^3 = 9y$$

$$\int_1^2 9y \, dy = \frac{9}{2} \, y^2 \Big|_1^2 = \frac{9}{2}(4-1) = \frac{27}{2}$$

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y) \mid a \leq x \leq b, \; c \leq y \leq d\}$ then,

$$\iint_R f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy$$

Vertically Simple Regions

A vertically simple region is a region s.t.,

$$D = \{(x,y) \mid a \leq x \leq b, \; g_1(x) \leq y \leq g_2(x)\}.$$

An integral over such a region is computed by,

$$\iint_D f(x,y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx$$

Horizontally Simple Region

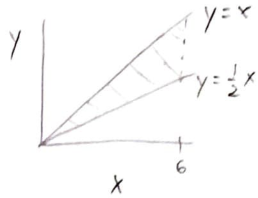
A region is horizontally simple if it can be written as,

$$D = \{(x,y) \mid c \leq y \leq d, \; h_1(y) \leq x \leq h_2(x)\}$$

An integral over such a region is computed by,

$$\iint_D f(x,y) \, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y) \, dy dx$$

Ex. 3

$\iint_R xy\,dA$ where $R =$ 

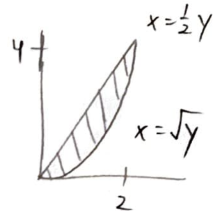
$x = \emptyset$ to $x = 6$ } vert simple
$y = \frac{1}{2}x$ to $y = x$

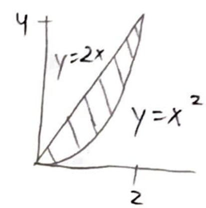
$$\int_0^6 \underbrace{\int_{\frac{1}{2}x}^{x} xy\,dy}\,dx =$$

$$x\int_{\frac{1}{2}x}^{x} y = \frac{x}{2}\,y^2\Big|_{\frac{1}{2}x}^{x} = \frac{x}{2}\left[x^2 - \frac{1}{4}x^2\right] = \frac{x}{2}\left[\frac{3}{4}x^2\right] = \frac{3}{8}x^3$$

$$\int_0^6 \frac{3}{8}x^3\,dx = \frac{3}{32}\,x^4\Big|_0^6 = \frac{3}{32}(1296) = \frac{243}{2}$$

Ex. 4

$\iint_D (x^2+y^2)\,dA$ where $D =$ 

or 

This region is both vert & horiz simple. It is thus a simple region. Such integrals can be computed both ways.